

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1893

EFFECT OF CENTRIFUGAL FORCE ON FLUTTER OF UNIFORM
CANTILEVER BEAM AT SUBSONIC SPEEDS WITH APPLICATION
TO COMPRESSOR AND TURBINE BLADES

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SUMMARY

An analysis was made of the effect of centrifugal force on the flutter of a uniform cantilever beam. Two methods of analysis were used. The first method was an approximate method in which the assumption was made that the flutter motion could be represented by using just two degrees of freedom. As these two degrees of freedom, the fundamental uncoupled modes in bending and torsion were used. In the second method, a solution was obtained by solving the exact equations of motion of a beam fluttering in a centrifugal-force field, with no assumption made concerning the actual mode shapes at flutter. This solution is exact provided that the oscillatory aerodynamic forces are exactly known. A comparison of the results obtained by both methods showed a negligible difference, which indicated that the approximate method using two degrees of freedom could be used with safety. The results of the analysis indicated that centrifugal force could be detrimental and could decrease the critical flutter speed. A beam that was stable at 0 rpm could become unstable at high rotative speeds. If, however, the flutter coefficient is kept below approximately 4, which is generally the case for compressor and turbine blades, there is little possibility of flutter occurring at low angles of attack.

INTRODUCTION

The study of vibrations of compressor and turbine blades has become of great importance as the use of axial-flow compressor and turbines has increased. Such studies involve resonant vibrations due to mechanical excitation from other parts of the machine or from pulsating air flows from previous rows of blades in addition to nonresonant aerodynamic excitation. The aerodynamic excitation involves a type of vibration that is self-sustained by the continual absorption of energy from the air stream. This vibration is called flutter.

Classical flutter is more specifically defined as a self-sustained oscillation due to the coupling of inertia forces, elastic forces, damping forces, and dynamic aerodynamic forces. This type of flutter usually occurs on airplane wings at low angles of attack when the air velocity reaches a certain value called the critical flutter speed. It is to be distinguished from a radically different type of flutter called stalling flutter, which occurs on airfoils at high angles of attack, such as propeller blades near the stall point (reference 1). Stalling flutter can be caused by an aerodynamic hysteresis effect, the negative slope of the lift curve, or excitation by a system of Kármán vortices (references 2 and 3).

Classical-flutter theory has been developed by many investigators (references 4 to 6) and the results have been found to agree with experiment. A direct application of this unmodified classical theory to analysis of compressor and turbine blades would seem to indicate little possibility of such blades fluttering at zero angle of attack. This improbability of flutter is due to the stiffness of compressor and turbine blades compared with that of airplane wings. Two significant factors, however, must be considered: the effect of centrifugal force and the effect of cascading. These effects may be either beneficial, that is, increase the critical flutter speed or detrimental, that is, decrease the critical flutter speed.

The classical approach to the flutter of a uniform wing is based on an important simplifying assumption, namely, the hypothesis of "semirigidity." This hypothesis implies that all flexural movements are in phase with one another and all torsional movements are in phase with one another. This hypothesis is equivalent to stating that the motion can be represented by a system of just two degrees of freedom. For the two degrees of freedom, two arbitrary modes in bending and torsion are then chosen that are assumed to be independent of the air velocity. The fundamental uncoupled modes in bending and torsion of the wing vibrating in vacuo are usually assumed for these two modes. This assumption is made in order to make the problem mathematically tractable and has given good results.

The exact solutions of the differential equations for the flutter of an elastic beam have been obtained (reference 7) for the special case of zero flexural stiffness and for the general case of finite flexural and torsional stiffnesses (reference 8). The results obtained show close agreement between the exact critical flutter speeds and the approximate values calculated on the basis of semirigid flutter theory.

An investigation was conducted at the NACA Lewis laboratory in order to obtain a rigorous solution to the problem of a uniform cantilever beam radially mounted fluttering in a centrifugal-force field and to investigate the effect of centrifugal force on the flutter of such a beam. Most of the calculations are made by the approximate method using the assumption that the motion can be represented by the two fundamental uncoupled modes in bending and torsion. The fundamental torsional mode is assumed to be unaffected by the centrifugal force and the fundamental bending mode is corrected for the centrifugal force, as indicated by Timoshenko (reference 9). The analysis then proceeds in the usual manner (reference 4). The rigorous solution, which is exact if the oscillatory aerodynamic forces and moments are exactly known, is obtained by the use of matrix methods and is then used to determine the error involved in approximating the true flutter motion by the uncoupled modes in bending and torsion when centrifugal forces are present.

SYMBOLS

The following symbols are used in the analysis:

$A(x)$	square matrix
a	coordinate of elastic axis measured from midchord, positive towards trailing edge, in units of half-chord
a_{ij}	elements of matrix
$a_1, b_1, c_1,$ b_2, c_2	functions of reduced frequency k and fundamental angular bending frequency ω_b or corrected fundamental angular bending frequency ω_b'
$a_y, \bar{a}_y, a_\theta, \bar{a}_\theta$	aerodynamic coefficients
b	half-chord used as reference unit length
$b_y, \bar{b}_y, b_\theta, \bar{b}_\theta$	aerodynamic coefficients
C	constant of integration
c	μr_g^2

EI_b	bending stiffness of beam
e	μr
F	function of k derived by Theodorsen (reference 4)
f_1, f_2	functions of ω_t/ω_b
G	function of k derived by Theodorsen (reference 4)
I	moment of inertia about elastic axis per unit span length
i	$\sqrt{-1}$
K	torsional stiffness of beam
k	reduced frequency, $\frac{\omega b}{v}$
L	oscillatory aerodynamic lift force per unit span
l	length of beam
M	oscillatory aerodynamic moment per unit span
M'	sum of moments acting on element of beam
M_b	bending moment
M_t	torsional moment
m	mass of airfoil per unit span
m_a	mass of cylinder of air (diameter of cylinder equal to airfoil chord) per unit span
R	distance from center of rotation to root of beam
r	location of center of gravity of airfoil measured from a, positive toward trailing edge in units of half- chord
r_g	radius of gyration referred to a, in units of half- chord

S	static mass unbalance of airfoil, $(m)(r)(b)$
T	axial tension
t	time
V	shear force
v	critical flutter speed
$v/\omega_t b$	flutter coefficient
x	dimensionless distance along length of beam, $\frac{x' - R}{l}$
x'	distance of element of beam from center of rotation
Y(x)	column matrix of unknowns
Y(0)	column matrix of constants
y	vertical displacement
y_0	maximum bending amplitude
$y_1(x)$	dimensionless bending mode
$y_2(x)$	dimensionless torsional mode
y_3, y_4, y_5, y_6	derivatives of y_1 and y_2 defined in text
β	proportionality constant, function of radial distance from center of rotation to root of blade and of blade taper
δ	phase angle by which torsion vibrational mode lags bending mode during flutter
θ	angle of torsional displacement
θ_0	maximum torsional amplitude
μ	ratio of mass of airfoil per unit span to mass of surrounding air cylinder per unit span, $\frac{m}{m_a}$

Ω	square matrix denoting matrizant of $A(x)$
Ω_{ij}	elements of Ω
ω	frequency of flutter vibration
ω_b	fundamental bending frequency of airfoil
ω_b'	fundamental bending frequency of airfoil corrected for centrifugal-force effect
ω_r	rotational frequency of beam
ω_t	fundamental torsional frequency of airfoil

ANALYSIS

The analysis of the effect of centrifugal force on the flutter of a uniform cantilever beam consists of two parts. The first part of the analysis uses the approximate solution to the flutter problem by the assumption of two degrees of freedom, namely, the uncoupled fundamental modes in bending and torsion. The fundamental bending mode is corrected for the centrifugal-force effect, whereas the fundamental torsional mode is assumed to be negligibly affected by centrifugal force. Most of the results presented herein were obtained by this method. In the second part of the analysis, the validity of the approximate solution is determined by solving the exact equations of motion for a beam fluttering in a centrifugal-force field. The results are then compared with those obtained by the approximate method. In the entire analysis, the oscillatory air forces and moments acting on the beam are assumed to be the same as those acting on a two-dimensional airfoil in a uniform air stream. Incompressible flow is assumed and structural frictions are assumed negligible.

Approximate Solution Assuming Two Degrees of Freedom

Flutter equations. - The approximate equations for dynamic equilibrium of the system using the uncoupled fundamental modes in bending and torsion are given in reference 4. In the notation used herein they become

$$\left. \begin{aligned} m\ddot{y} + S\ddot{\theta} + m(\omega_b')^2 y - L &= 0 \\ I\ddot{\theta} + S\ddot{y} + I\omega_t^2 \theta - M &= 0 \end{aligned} \right\} \quad (1)$$

The dots indicate differentiation with respect to time.

The oscillatory aerodynamic lift L and moment M are given by

$$\left. \begin{aligned} L &= -m_a \omega^2 \left[(a_y + i\bar{a}_y)y + (a_\theta + i\bar{a}_\theta)b\theta \right] \\ M &= -m_a \omega^2 b \left[(b_y + i\bar{b}_y)y + (b_\theta + i\bar{b}_\theta)b\theta \right] \end{aligned} \right\} \quad (2)$$

where

$$\left. \begin{aligned} a_y &= -\left(1 + \frac{2}{k}G\right) \\ \bar{a}_y &= \frac{2}{k}F \\ a_\theta &= a + \frac{2}{k^2}F - \frac{2}{k}\left(\frac{1}{2} - a\right)G \\ \bar{a}_\theta &= \frac{1}{k} + \frac{2}{k^2}G + \frac{2}{k}\left(\frac{1}{2} - a\right)F \\ b_y &= a + \frac{2}{k}\left(a + \frac{1}{2}\right)G \\ \bar{b}_y &= -\frac{2}{k}\left(a + \frac{1}{2}\right)F \\ b_\theta &= -\left[\frac{1}{8} + a^2 + \frac{2}{k^2}\left(a + \frac{1}{2}\right)F - \frac{2}{k}\left(\frac{1}{4} - a^2\right)G\right] \\ \bar{b}_\theta &= -\left[\frac{2}{k^2}\left(a + \frac{1}{2}\right)G + \frac{2}{k}\left(\frac{1}{4} - a^2\right)F - \frac{1}{k}\left(\frac{1}{2} - a\right)\right] \end{aligned} \right\} \quad (3)$$

where F and G are functions of only the reduced frequency k given in reference 4.

Let

$$\left. \begin{aligned} y &= y_0 e^{i\omega t} \\ \theta &= \theta_0 e^{i(\omega t - \delta)} \end{aligned} \right\} \quad (4)$$

then

$$\left. \begin{aligned} \ddot{y} &= -\omega^2 y \\ \ddot{\theta} &= -\omega^2 \theta \end{aligned} \right\} \quad (5)$$

By combining equations (1), (2), and (5) and dividing through by $m_a \omega^2$

$$\left. \begin{aligned} \left[a_y + i\bar{a}_y + \mu \left(\frac{\omega_b'^2}{\omega^2} - 1 \right) \right] y + (a_\theta + i\bar{a}_\theta - e) b \theta &= 0 \\ (b_y + i\bar{b}_y - e) b y + \left[b_\theta + i\bar{b}_\theta + c \left(\frac{\omega_t^2}{\omega^2} - 1 \right) \right] b^2 \theta &= 0 \end{aligned} \right\} \quad (6)$$

Equations (6) have solutions other than $y = \theta = 0$ if and only if the determinant of the coefficients of y and θ vanishes. If this determinant is set equal to zero, real and imaginary parts are separated, and the equation is rearranged,

$$\left. \begin{aligned} \left(\frac{\omega}{\omega_t} \right)^2 &= \frac{b_2}{c_2} \\ a_1 \left(\frac{\omega}{\omega_t} \right)^4 + b_1 \left(\frac{\omega}{\omega_t} \right)^2 + c_1 &= 0 \end{aligned} \right\} \quad (7)$$

and

where

$$b_2 = c\bar{a}_y + \mu \left(\frac{\omega_b'}{\omega_t} \right)^2 \bar{b}_\theta$$

$$c_2 = c\bar{a}_y + \mu \bar{b}_\theta + \bar{a}_\theta (b_y - e) + \bar{b}_y (a_\theta - e) - b_\theta \bar{a}_y - a_y \bar{b}_\theta$$

$$a_1 = (c - b_\theta) (\mu - a_y) + \bar{b}_y \bar{a}_\theta - \bar{b}_\theta \bar{a}_y - (a_\theta - e) (b_y - e)$$

$$b_1 = ca_y + \mu \left(\frac{\omega_b'}{\omega_t} \right)^2 b_\theta - c\mu \left[\left(\frac{\omega_b'}{\omega_t} \right)^2 + 1 \right]$$

$$c_1 = c\mu \left(\frac{\omega_b'}{\omega_t} \right)^2$$

For a given airfoil, a_1, b_1, c_1, b_2 , and c_2 are functions only of k and the fundamental bending frequency corrected for centrifugal force effects ω_b' , which is, in turn, a function of the rotative speed. For a given rotative speed, a, b , and c are functions of only the reduced frequency. Equations (7) can then be graphically solved for k and the ratio of flutter frequency to fundamental torsional frequency ω/ω_t . In the calculations performed herein, a value was assumed for k and the corresponding values of ω/ω_t and ω_b'/ω_t were calculated. This procedure obviated the necessity for a graphical solution.

Bending frequency. - The effect of centrifugal force on fundamental bending frequency ω_b was obtained from reference 9.

$$(\omega_b')^2 = \omega_b^2 + \beta^2 \omega_r^2 \quad (8)$$

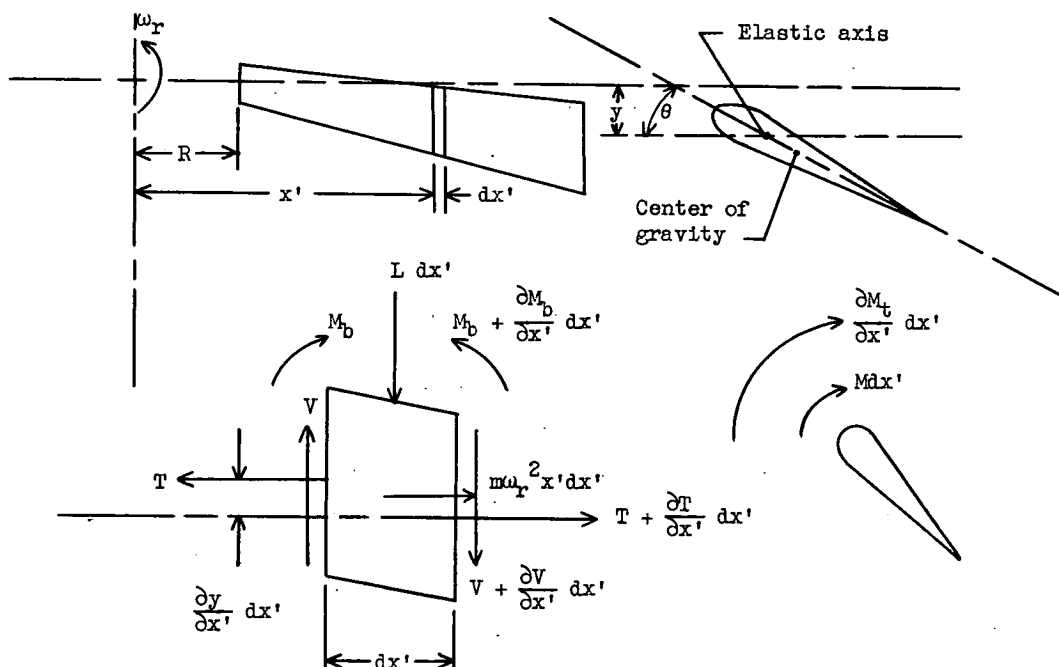
where ω_b is the fundamental bending frequency without rotation and β is a function of the ratio of blade length to radial distance from center of rotation to root of blade and of blade taper. By combining equations (7) and (8), k and the flutter frequency ω can be obtained as a function of the rotative speed. The critical flutter speed v can then be obtained from the relation

$$v = \frac{\omega b}{k} \quad (9)$$

Actually, it was thought more useful to obtain the dimensionless parameter $v/\omega_t b$ as a function of the dimensionless parameter ω_r/ω_b .

Exact Solution for Beam Fluttering in Centrifugal-Force Field

Equations of dynamic equilibrium. - The equations of motion for the beam are obtained from the equations for the equilibrium of forces acting on an infinitesimal element of the beam.



The external forces and moments acting on an element of the beam as shown consist of transverse shearing forces and torsional moments that are transmitted from one portion of the beam to the next, plus the centrifugal force and moment and the aerodynamic force and moment. For dynamic equilibrium to exist, these external forces and moments must be balanced by the inertia forces and moments. The following equations of equilibrium can therefore be given:

For the equilibrium of moments in the x - y plane,

$$M' = \frac{\partial M_b}{\partial x'} dx' - V dx' - \frac{\partial V}{\partial x'} \frac{(dx')^2}{2} + T \frac{\partial y}{\partial x'} dx' + \frac{\partial T}{\partial x'} \frac{\partial y}{\partial x'} \frac{(dx')^2}{2} = 0$$

which upon neglecting terms of higher-order differentials becomes

$$\frac{\partial M_b}{\partial x'} - V + T \frac{\partial y}{\partial x'} = 0 \quad (10)$$

For the equilibrium of forces in the x' direction,

$$\frac{\partial T}{\partial x'} dx' + m\omega_r^2 x' dx' = 0$$

or

$$\frac{\partial T}{\partial x'} = -m\omega_r^2 x' \quad (11)$$

which upon integration becomes

$$T = -\frac{m\omega_r^2}{2} (x')^2 + C$$

and inasmuch as $T = 0$ when $x' = l + R$,

$$C = \frac{m\omega_r^2}{2} (l + R)^2$$

therefore

$$T = \frac{m\omega_r^2}{2} \left[(l + R)^2 - (x')^2 \right] \quad (12)$$

For the equilibrium of forces in the y direction,

$$\frac{\partial V}{\partial x'} dx' + L dx' - m\ddot{y} dx' - S\ddot{\theta} dx' = 0$$

or

$$\frac{\partial V}{\partial x'} + L - m\ddot{y} - \ddot{S}\theta = 0 \quad (13)$$

From equation (10),

$$\frac{\partial V}{\partial x'} = \frac{\partial^2 M_b}{\partial (x')^2} + T \frac{\partial^2 y}{\partial (x')^2} + \frac{\partial T}{\partial x'} \frac{\partial y}{\partial x'} \quad (14)$$

If equations (11), (12), (13), and (14) are combined,

$$\begin{aligned} \frac{\partial^2 M_b}{\partial (x')^2} + \frac{m\omega_r^2}{2} \left[(l+R)^2 - (x')^2 \right] \frac{\partial^2 y}{\partial (x')^2} - \\ m\omega_r^2 x' \frac{\partial y}{\partial x'} + L - m\ddot{y} - \ddot{S}\theta = 0 \end{aligned} \quad (15)$$

For the equilibrium of torsional moments,

$$\frac{\partial M_t}{\partial x'} + M - S\omega_r^2 x' \frac{\partial y}{\partial x'} - I\ddot{\theta} - S\ddot{y} = 0 \quad (16)$$

From elementary elastic theory,

$$\frac{\partial^2 M_b}{\partial (x')^2} = -EI_b \frac{\partial^4 y}{\partial (x')^4} \quad (17)$$

and

$$\frac{\partial M_t}{\partial x'} = K \frac{\partial^2 \theta}{\partial (x')^2}$$

equations (15) and (16) now become

$$\left. \begin{aligned}
 m\ddot{y} + S\ddot{\theta} + EI_b \frac{\partial^4 y}{\partial (x')^4} - \frac{m\omega_r^2}{2} \left[(l + R)^2 - (x')^2 \right] \frac{\partial^2 y}{\partial (x')^2} + \\
 m\omega_r^2 x' \frac{\partial y}{\partial x'} - L = 0 \\
 S\ddot{y} + S\omega_r^2 x' \frac{\partial y}{\partial x'} + I\ddot{\theta} - K \frac{\partial^2 \theta}{\partial (x')^2} - M = 0
 \end{aligned} \right\} \quad (18)$$

When the values of L and M as given by equation (2) are substituted,

$$\left. \begin{aligned}
 m\ddot{y} + S\ddot{\theta} + EI_b \frac{\partial^4 y}{\partial (x')^4} - \frac{m\omega_r^2}{2} \left[(l + R)^2 - (x')^2 \right] \frac{\partial^2 y}{\partial (x')^2} + \\
 m\omega_r^2 x' \frac{\partial y}{\partial x'} + \omega_m^2 (a_y + i\bar{a}_y) y + \omega_m^2 b(a_\theta + i\bar{a}_\theta)\theta = 0 \\
 S\ddot{y} + S\omega_r^2 x' \frac{\partial y}{\partial x'} + I\ddot{\theta} - K \frac{\partial^2 \theta}{\partial (x')^2} + \omega_m^2 b(b_y + i\bar{b}_y)y + \\
 \omega_m^2 b^2(b_\theta + i\bar{b}_\theta)\theta = 0
 \end{aligned} \right\} \quad (19)$$

The following substitution of variables is convenient:

$$x = \frac{x' - R}{l} \quad (20)$$

Then equation (19) becomes

$$\left. \begin{aligned}
 m\ddot{y} + S\ddot{\theta} + \frac{EI_b}{l^4} \frac{\partial^4 y}{\partial x^4} - \frac{m\omega_r^2}{2} \left[\left(1 + \frac{R}{l}\right)^2 - \left(x + \frac{R}{l}\right)^2 \right] \frac{\partial^2 y}{\partial x^2} + \\
 m\omega_r^2 \left(x + \frac{R}{l}\right) \frac{\partial y}{\partial x} + \omega^2 m_a (a_y + i \bar{a}_y) y + \omega^2 m_a b (a_\theta + i \bar{a}_\theta) \theta = 0 \\
 \text{and} \\
 S\ddot{y} + S\omega_r^2 \left(x + \frac{R}{l}\right) \frac{\partial y}{\partial x} + I\ddot{\theta} - \frac{K}{l^2} \frac{\partial^2 \theta}{\partial x^2} + \omega^2 m_a b (b_y + i \bar{b}_y) y + \\
 \omega^2 m_a b^2 (b_\theta + i \bar{b}_\theta) \theta = 0
 \end{aligned} \right\} (21)$$

These are the equations of motion.

Solution of equations of motion. -

Let

$$\left. \begin{aligned}
 y &= b y_1(x) e^{i\omega t} \\
 \theta &= y_2'(x) e^{i(\omega t - \delta)}
 \end{aligned} \right\} (22)$$

and let

$$y_2(x) = e^{-i\delta} y_2'(x)$$

then

$$\theta = y_2(x) e^{i\omega t}$$

If these expressions are substituted into equations (21) and the first equation divided by $\omega^2 m_a b e^{i\omega t}$ and the second equation by $\omega^2 m_a b^2 e^{i\omega t}$, the following dimensionless equations are obtained:

$$\left. \begin{aligned}
 & \left[(a_y - \mu) + i\bar{a}_y \right] y_1 + \frac{EI_b}{\omega^2 m_a l^4} \frac{d^4 y_1}{dx^4} - \frac{\mu \omega^2 r^2}{2\omega^2} \left[\left(1 + \frac{R}{l}\right)^2 - \left(x + \frac{R}{l}\right)^2 \right] \frac{d^2 y_1}{dx^2} + \\
 & \frac{\mu \omega^2 r^2}{\omega^2} \left(x + \frac{R}{l}\right) \frac{dy_1}{dx} + \left[(a_\theta - \mu r) + i\bar{a}_\theta \right] y_2 = 0 \\
 & \left[(b_y - \mu r) + i\bar{b}_y \right] y_1 + \mu r \frac{\omega r^2}{\omega^2} \left(x + \frac{R}{l}\right) \frac{dy_1}{dx} + \left[(b_\theta - \mu r_g^2) + i\bar{b}_\theta \right] y_2 - \\
 & \frac{K}{\omega^2 m_a l^2 b^2} \frac{d^2 y_2}{dx^2} = 0
 \end{aligned} \right\} (23)$$

The first of equations (23) is a fourth-order linear differential equation with variable coefficients and the second equation is a second-order linear equation with variable coefficients. No simple methods exist for solving such a pair of simultaneous equations. The method used herein is to reduce the equations to a system of first-order equations and then apply the Peano-Baker method of integration (reference 10).

Let

$$\left. \begin{aligned}
 \frac{dy_1}{dx} &= y_3 \\
 \frac{d^2 y_1}{dx^2} &= \frac{dy_3}{dx} = y_4 \\
 \frac{d^3 y_1}{dx^3} &= \frac{dy_4}{dx} = y_6 \\
 \frac{dy_2}{dx} &= y_5
 \end{aligned} \right\} (24)$$

Then equations (23) become

$$\left. \begin{aligned} \frac{dy_6}{dx} &= a_{61}y_1 + a_{62}y_2 + a_{63}y_3 + a_{64}y_4 \\ \frac{dy_5}{dx} &= a_{51}y_1 + a_{52}y_2 + a_{53}y_3 \end{aligned} \right\} \quad (25)$$

where

$$\begin{aligned} a_{61} &= \frac{\mu - a_y - i\bar{a}_y}{\frac{EI_b}{\omega^2 m_a l^4}} & a_{51} &= \frac{b_y - \mu r + i\bar{b}_y}{\frac{K}{\omega^2 m_a l^2 b^2}} \\ a_{62} &= \frac{\mu r - a_\theta - i\bar{a}_\theta}{\frac{EI_b}{\omega^2 m_a l^4}} & a_{52} &= \frac{b_\theta - \mu r_g^2 + i\bar{b}_\theta}{\frac{K}{\omega^2 m_a l^2 b^2}} \\ a_{63} &= \frac{-\mu \omega r^2}{\frac{EI_b}{m_a l^4}} \left(x + \frac{R}{l}\right) & a_{53} &= \frac{\mu r \omega r^2}{\frac{K}{m_a l^2 b^2}} \left(x + \frac{R}{l}\right) \\ a_{64} &= \frac{\mu \omega r^2}{\frac{2EI_b}{m_a l^4}} \left[\left(1 + \frac{R}{l}\right)^2 - \left(x + \frac{R}{l}\right)^2 \right] \end{aligned}$$

The system of six first-order equations (24) and (25) can now be replaced by the single first-order matrix equation:

$$\frac{dY(x)}{dx} = A(x) Y(x) \quad (26)$$

where $Y(x)$ is the column matrix

$$Y(x) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

and $A(x)$ is the square matrix:

$$A(x) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_{51} & a_{52} & a_{53} & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & 0 & 0 \end{bmatrix}$$

The elements a_{ij} of the matrix $A(x)$ are not all constant. The solution of matrix equation (26) can now be set down as

$$Y(x) = \Omega_0^x Y(0) \quad (27)$$

where Ω_0^x is the matrizant of $A(x)$ and is given by

$$\begin{aligned}
\Omega_0^x &= I + \int_0^x A \, dx + \int_0^x A \int_0^{x_1} A \, dx_1 \, dx + \int_0^x A \int_0^{x_1} A \int_0^{x_2} A \, dx_2 \, dx_1 \, dx + \\
&\quad \dots + \int_0^x A \int_0^{x_1} A \dots \int_0^{x_n} A \, dx_n \, dx_{n-1} \dots dx_1 \, dx + \dots \\
&= I + \sum_{n=0}^{\infty} \int_0^x A \int_0^{x_1} A \dots \int_0^{x_{n-1}} A \int_0^{x_n} A \, dx_n \, dx_{n-1} \dots dx_1 \, dx
\end{aligned}$$

where $x_0 = x$ and I is the unit matrix of order six.

Boundary conditions. - The boundary conditions to be satisfied for the case of a cantilever beam are

$$\left. \begin{aligned}
x = 0 \quad y_1(0) = y_2(0) = y_3(0) = 0 \\
x = 1 \quad y_4(1) = y_5(1) = y_6(1) = 0
\end{aligned} \right\} \quad (29)$$

From the conditions at $x = 0$, the column matrix $Y(0)$ becomes

$$Y(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{bmatrix} \quad (30)$$

For $x = 1$, equation (27) gives

$$\left. \begin{aligned}
y_4(1) = 0 &= \Omega_{44} y_4(0) + \Omega_{45} y_5(0) + \Omega_{46} y_6(0) \\
y_5(1) = 0 &= \Omega_{54} y_4(0) + \Omega_{55} y_5(0) + \Omega_{56} y_6(0) \\
y_6(1) = 0 &= \Omega_{64} y_4(0) + \Omega_{65} y_5(0) + \Omega_{66} y_6(0)
\end{aligned} \right\} \quad (31)$$

where Ω_{ij} is the element of row i and column j of the matrizant Ω_0^1 .

Critical flutter speed and frequency. - The critical flutter speed and frequency can now be obtained by means of equations (31). Equations (31) have solutions other than $y_4(0) = y_5(0) = y_6(0) = 0$ if and only if the determinant of the coefficients vanishes. Therefore,

$$\begin{vmatrix} \Omega_{44} & \Omega_{45} & \Omega_{46} \\ \Omega_{54} & \Omega_{55} & \Omega_{56} \\ \Omega_{64} & \Omega_{65} & \Omega_{66} \end{vmatrix} = 0 \quad (32)$$

or

$$\begin{vmatrix} \Omega_{44} & \Omega_{55} & \Omega_{66} \\ \Omega_{55} & \Omega_{64} & \Omega_{45} \\ \Omega_{64} & \Omega_{45} & \Omega_{54} \end{vmatrix} + \begin{vmatrix} \Omega_{45} & \Omega_{56} & \Omega_{64} \\ \Omega_{56} & \Omega_{66} & \Omega_{44} \\ \Omega_{64} & \Omega_{44} & \Omega_{56} \end{vmatrix} + \begin{vmatrix} \Omega_{46} & \Omega_{54} & \Omega_{65} \\ \Omega_{54} & \Omega_{65} & \Omega_{46} \\ \Omega_{65} & \Omega_{46} & \Omega_{54} \end{vmatrix} = 0 \quad (33)$$

The elements Ω_{ij} are complex and for a given airfoil at a given rotative speed are functions only of ω and k . The real and imaginary parts of equation (33) can then be separated and the resultant two equations simultaneously solved for ω and k . The calculations are very laborious and have been made herein for a given airfoil for two values of rotative speed.

RESULTS AND DISCUSSION

Equations (7) were solved for k and ω/ω_t as functions of the ratio of torsional frequency to bending frequency corrected for centrifugal-force effect ω_t/ω_b' ; the results are shown in figure 1. Calculations were made for values of center-of-gravity position r of 0.1, 0.2, and 0.3. These values cover the practical range. At $r = 0.1$, mass-ratio values μ of 100, 300, 700, and 1000 were used. At the two other values of r , values of μ of 700 and 1000 were used. The calculations were made at such high values of μ because the mass ratios of compressor and turbine blades are much higher than for airplane wings. The value of the radius of gyration r was taken as 0.5 for all calculations

and the value of elastic-axis position measured from midchord a as -0.4 . Changes in a have a comparatively small effect (if the elastic axis is not very close to the gravity axis) and this one value in the practical range was chosen to reduce the number of calculations.

Examination of figure 1 shows that the effect of increasing μ is to decrease both ω/ω_t and k , the decrease in k generally being greater. Inasmuch as v is proportional to the ratio of ω/ω_t to k , it will generally increase with increasing μ . This effect becomes smaller at larger values of μ .

The effect of increasing r is to decrease the value of ω/ω_t for high values of ω_t/ω_b' , whereas the values of ω/ω_t are left relatively unchanged at the low values of ω_t/ω_b' . The values of k are either unaffected or increase at the higher values of ω_t/ω_b' but are decreased at the lower values of ω_t/ω_b' where the peak occurs. This relation would indicate that the effect of increasing r is to decrease v except near the point where ω_t/ω_b' equals 1, where the peak occurs in the k curve.

In order to determine the effect of centrifugal force on v , figure 1 must be more closely examined. Low values of the abscissa ω_t/ω_b' correspond to high values of rotative speed and high values of the abscissa correspond to low values of the rotative speed. This relation can be seen if equation (8) is rewritten in the form:

$$\begin{aligned} \left(\frac{\omega_t}{\omega_b'}\right)^2 &= \frac{\omega_t^2}{\omega_b^2 + \beta^2 \omega_r^2} \\ &= \frac{\left(\frac{\omega_t}{\omega_b}\right)^2}{1 + \beta^2 \left(\frac{\omega_r}{\omega_b}\right)^2} \end{aligned} \quad (34)$$

For a given airfoil, ω_t/ω_b' will therefore decrease as ω_r/ω_b increases. From figure 1 for values of ω_t/ω_b' less than approximately 1, the value of k will decrease and the value of ω/ω_t will increase as the value of the rotative frequency ω_r increases (or as ω_t/ω_b' decreases). At values of ω_t/ω_b' greater than approximately 1, both k and ω/ω_t increase as ω_r increases (or as ω_t/ω_b' decreases). If k increases at a faster rate than ω/ω_t in any part of the region in which both ω/ω_t and k are increasing, a minimum flutter speed would exist at a certain rotative speed. This relation can be seen as follows: The critical flutter speed can be expressed as

$$v = \frac{b \frac{\omega}{\omega_t} \omega_t}{k} \quad (35)$$

Values of v will therefore increase with increasing rotational speed for values of less than approximately 1 and will decrease with increasing rotative speed for values of ω_t/ω_b' greater than approximately 1. The existence of a minimum v is therefore indicated.

The foregoing relations can perhaps be more clearly seen if equation (34) is used with figure 1 to plot directly k and ω/ω_t as a function of the rotative speed. This relation is shown plotted in figures 2 and 3 for values of ω_t/ω_b ranging from 1 to 10. The value of μ is 700 and the constant β is set at 1.0, which corresponds to a nontapered blade with a ratio of root radius to blade length R/l of 0. As seen from figure 3 as ω_r/ω_b increases or as the rotative speed increases, the frequency ratio ω/ω_t increases and from figure 2 the reduced frequency k in general at first increases and then decreases. If the rate of increase of k in any part of the region where it is increasing is greater than the rate of increase of ω/ω_t , then from equation (35) a minimum value of v can be expected.

In order to illustrate the previous conclusions, v was calculated from figure 1 and equation (35) for different values of

ω_t/ω_b and different values of R/l . The results are shown in dimensionless form in figures 4 to 6. The abscissa is ω_r/ω_b and the ordinate is the dimensionless flutter coefficient $v/\omega_t b$. The values of R/l used are 0, 4, and 10. The corresponding values of β taken from reference 9 are 1.00, 2.62, and 3.95.

The values of $v/\omega_t b$, and for a given blade v , decrease to a minimum as ω_r/ω_b or the rotative speed increases (figs. 4 to 6). For a given value of μ and r , this minimum value of $v/\omega_t b$ is a constant for all values of ω_t/ω_b . The value of ω_r/ω_b at which this minimum occurs, however, increases as the ratio ω_t/ω_b increases. This decrease in v as the rotative speed is increased is caused by the centrifugal force causing only ω_b to increase. Although increasing both ω_b and ω_t will increase v , increasing ω_b alone may decrease v (figs. 4 to 6 and reference 11). Values of v may be decreased by the centrifugal-force effect as much as 65 percent.

The effect of R/l is in the same direction as the effect of rotative speed. Increasing R/l increases the centrifugal-force effect. The effect is therefore the same as increasing the rotative speed. The minimum points on the curves are therefore shifted to lower values of the rotative speed as can be seen in the figures.

The lowest value of the flutter coefficient $v/\omega_t b$ obtained in these calculations was approximately 4.3. In the practical range of parameters for current compressor and turbine blades, this value is probably close to the lowest value that is to be expected. If the blades are so designed that the operating $v/\omega_t b$ is lower than approximately 4, little possibility exists of classical flutter occurring. When sufficient knowledge becomes available on the effects of cascading on the oscillatory aerodynamic forces and moments, the previous results may have to be modified. The previous results apply to low angles of attack. With blades operating at high angles of attack near the stall point, v may be considerably reduced.

In order to determine the possible error in the previous calculations because of approximating the flutter motion by the assumption of uncoupled modes in bending and torsion, an exact solution

was obtained for two points by means of equation (33). The results are shown in figure 7 where k and ω/ω_b are plotted against ω_r/ω_b . The curves were obtained by the approximate method used in obtaining figures 1 to 6. The circles represent the points obtained by the exact solution. The difference in the results obtained by the approximate solution and by the much more laborious exact method is negligible. The point at $\omega_r/\omega_b = 0$ verifies the conclusions of reference 8 that the use of the uncoupled modes in bending and torsion as degrees of freedom gives results that differ negligibly from the exact solution for a simple cantilever beam. The point at $\omega_r/\omega_b = 1$ indicates that the same conclusion can be drawn for a cantilever beam in a centrifugal-force field provided that the fundamental bending mode is corrected for centrifugal-force effect.

CONCLUSIONS

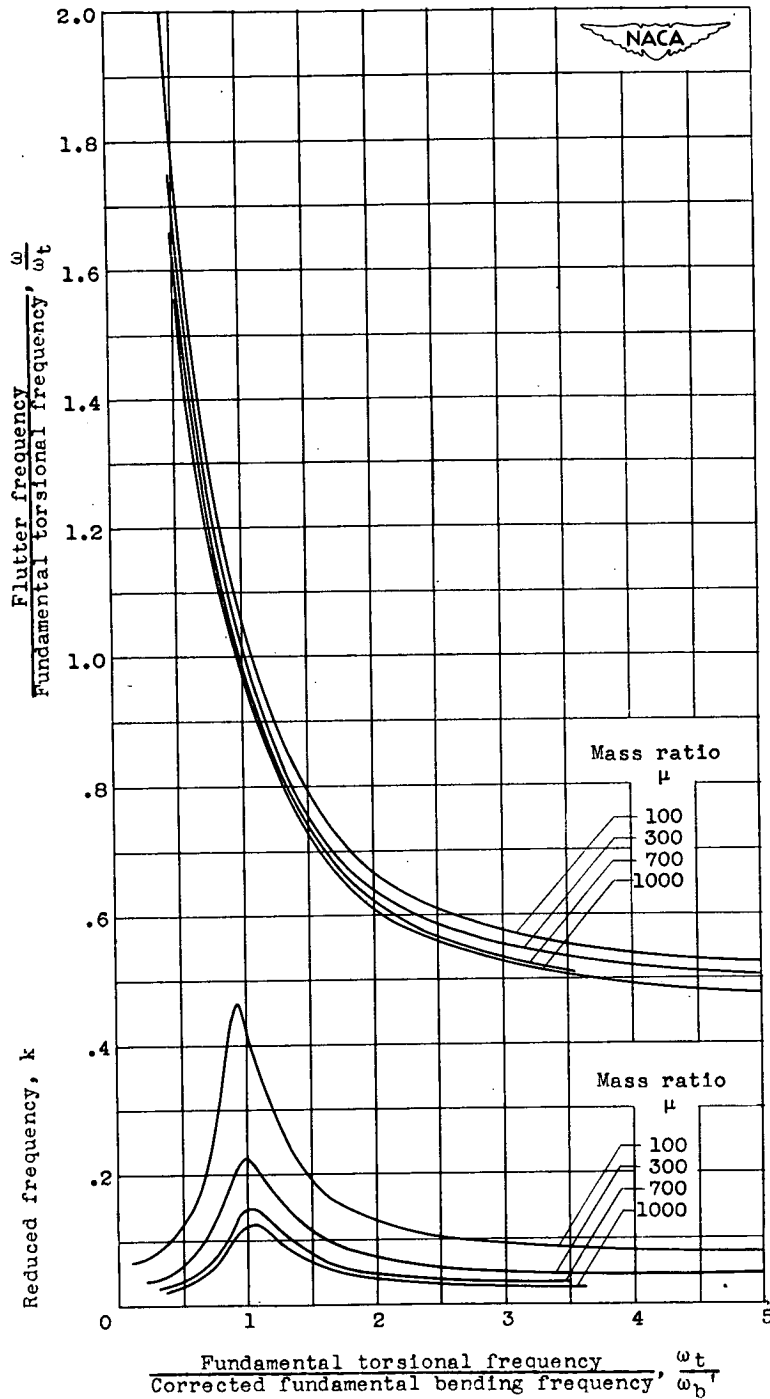
On the basis of calculations carried out by exact and approximate methods on the effect of centrifugal force on the flutter of a uniform cantilever beam it is concluded that:

1. The approximate method of solution, using as two degrees of freedom the uncoupled fundamental modes in bending and torsion in vacuo, gave results that differed negligibly from those obtained by exact methods, which indicated that the approximate method can be used with safety.
2. The effect of centrifugal force could be detrimental and could decrease the critical flutter speed for the range of parameters considered as much as 65 percent under certain conditions. A beam that was stable at 0 rpm could become unstable at high rotative speeds.
3. In the practical range of blade parameters, ignoring the effect of cascading, there would be little probability of compressor and turbine blades fluttering at low angles of attack if the flutter coefficient was kept below approximately 4.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, April 6, 1949.

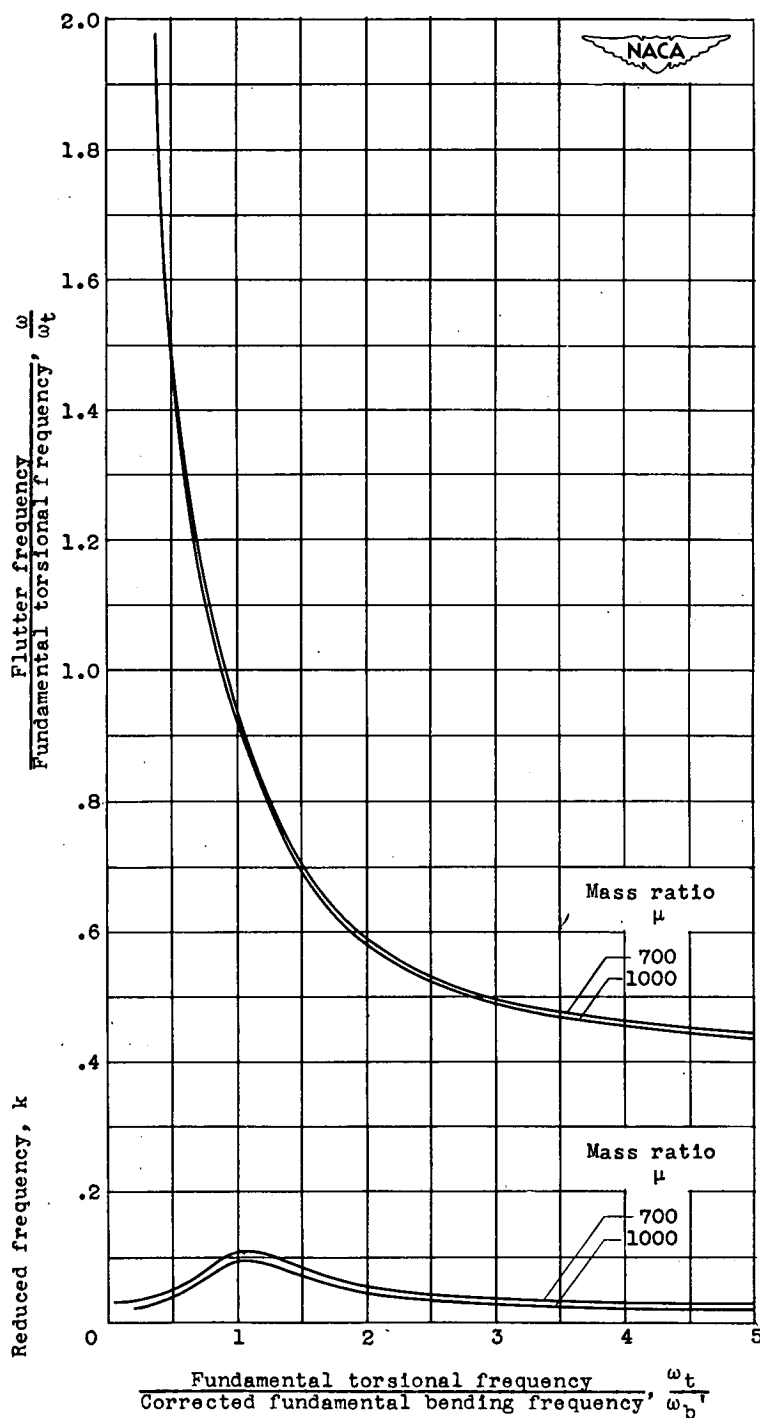
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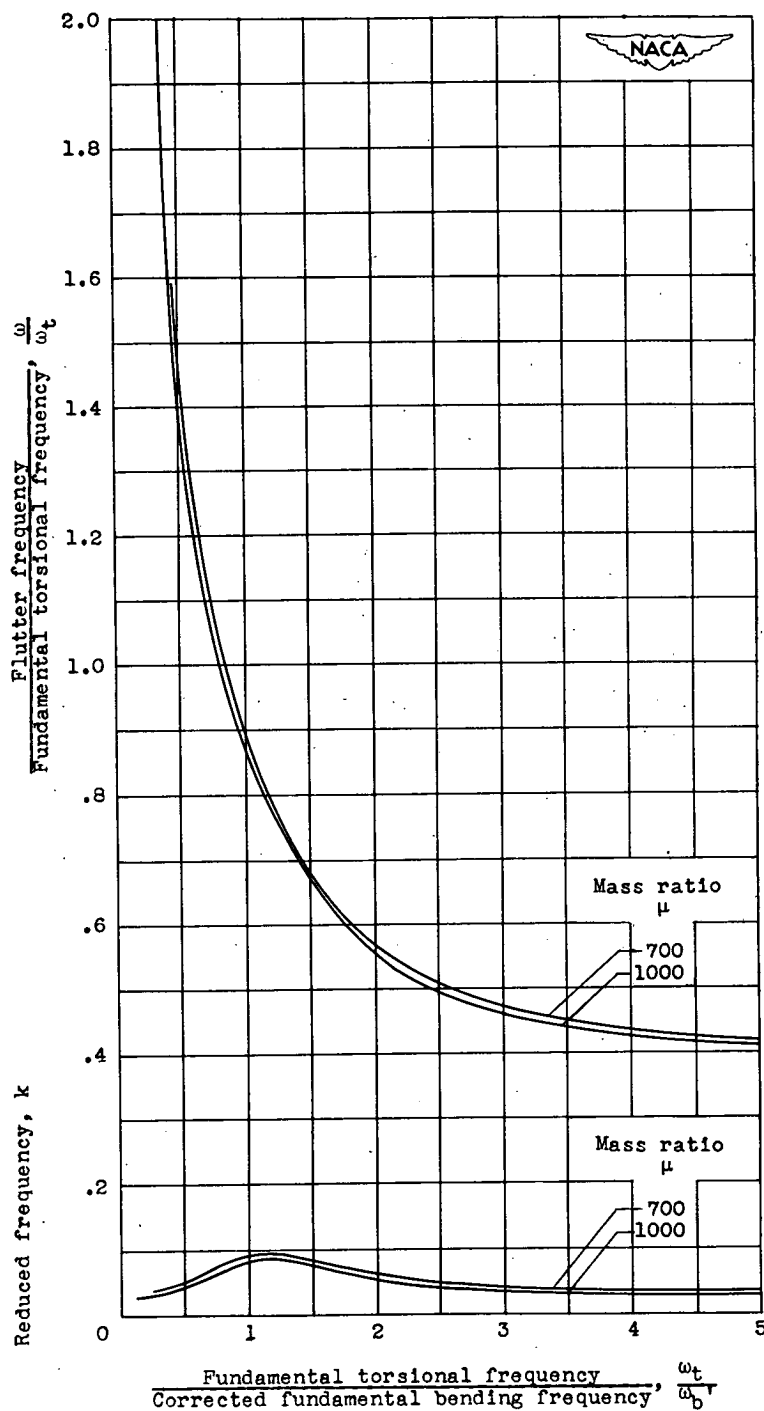
(a) Center-of-gravity position measured from position $a, r, 0.1$; elastic-axis position measured from midchord $a, -0.4$; radius of gyration $r_g, 0.5$.

Figure 1. - Variation of reduced frequency and ratio of flutter frequency to fundamental torsional frequency with ratio of fundamental torsional frequency to corrected fundamental bending frequency for various mass ratios.



(b) Center-of-gravity position measured from position $a, r, 0.2$; elastic-axis position measured from midchord $a, -0.4$; radius of gyration $r_g, 0.5$.

Figure 1. - Continued. Variation of reduced frequency and ratio of flutter frequency to fundamental torsional frequency with ratio of fundamental torsional frequency to corrected fundamental bending frequency for various mass ratios.



(c) Center-of-gravity position measured from position a , r , 0.3; elastic-axis position measured from midchord a , -0.4; radius of gyration r_g , 0.5.

Figure 1. - Concluded. Variation of reduced frequency and ratio of flutter frequency to fundamental torsional frequency with ratio of fundamental torsional frequency to corrected fundamental bending frequency for various mass ratios.

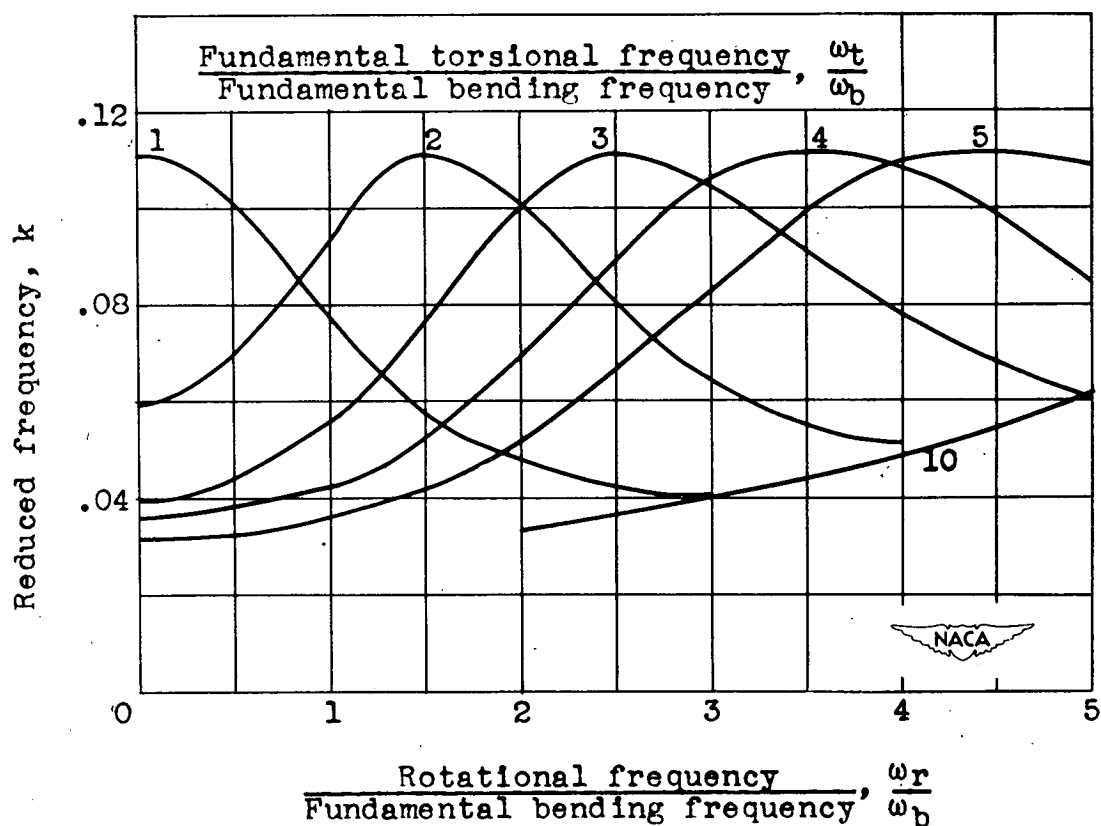


Figure 2. - Variation of reduced frequency with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Mass ratio μ , 700; center-of-gravity position measured from position a, r , 0.2; root-radius-to-length ratio R/l , 0.

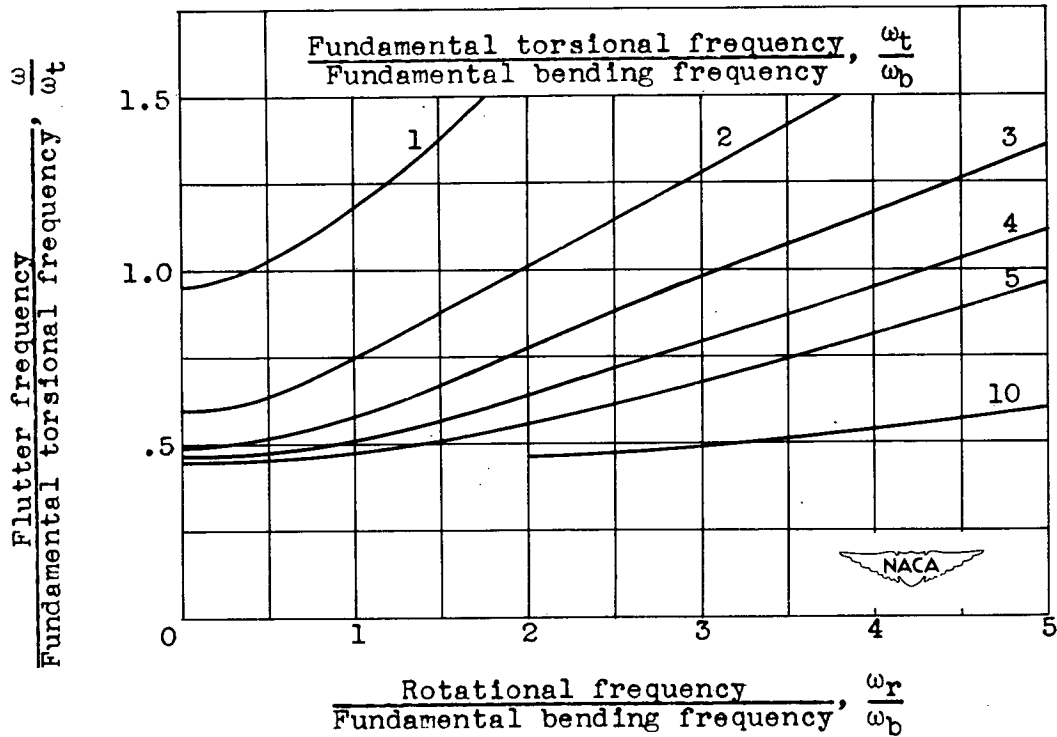


Figure 3. - Variation of ratio of flutter frequency to fundamental torsional frequency with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Mass ratio μ , 700; center-of-gravity position measured from position a , r , 0.2; root-radius-to-length ratio R/l , 0.

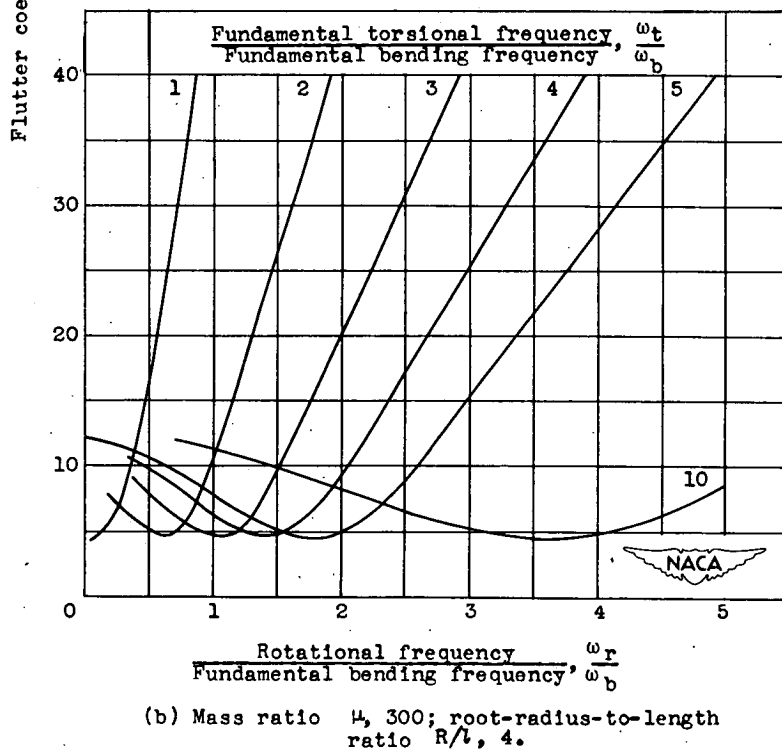
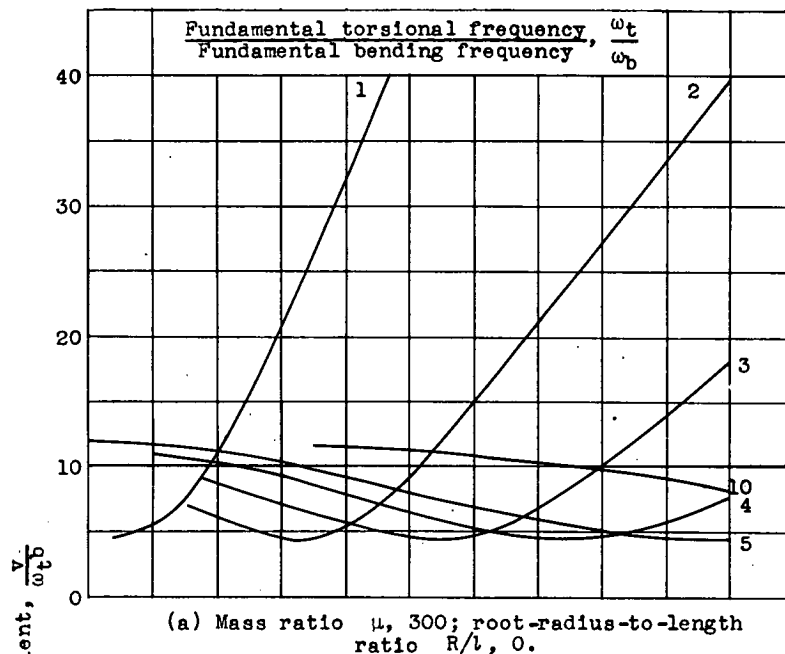


Figure 4. - Variation of flutter coefficient with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Center-of-gravity position measured from position a, r , 0.1; elastic-axis position measured from midchord a, -0.4 ; radius of gyration r_g , 0.5.

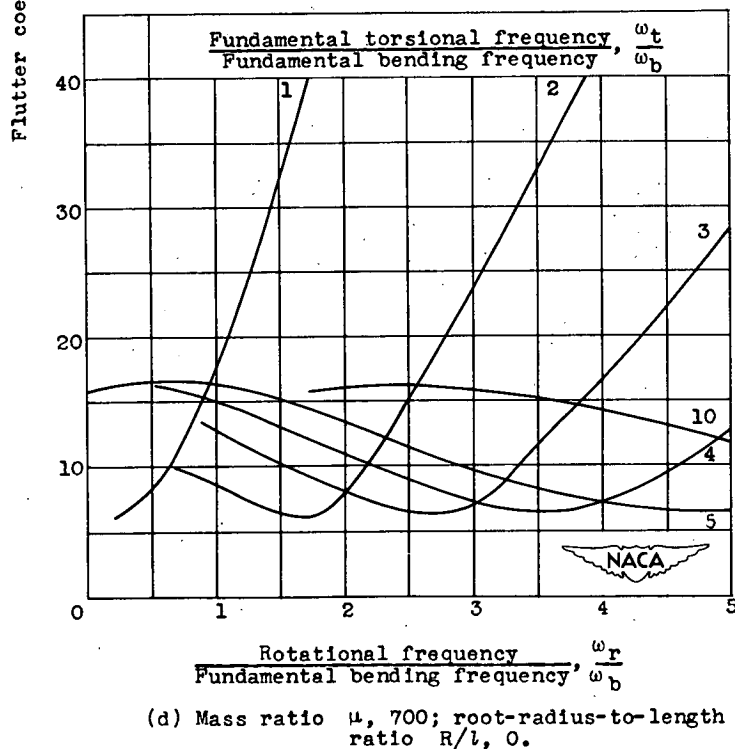
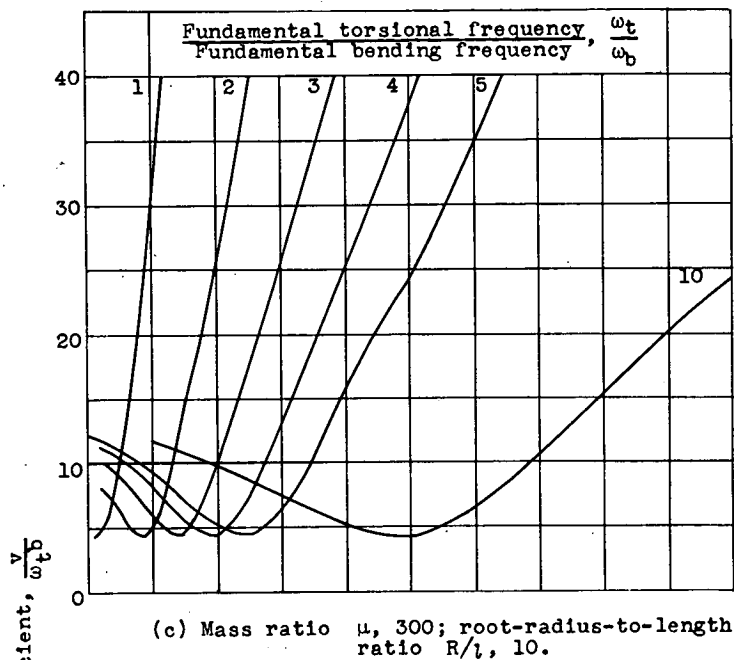


Figure 4. - Continued. Variation of flutter coefficient with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Center-of-gravity position measured from position a, r , 0.1; elastic-axis position measured from midchord a, -0.4; radius of gyration r_g , 0.5.

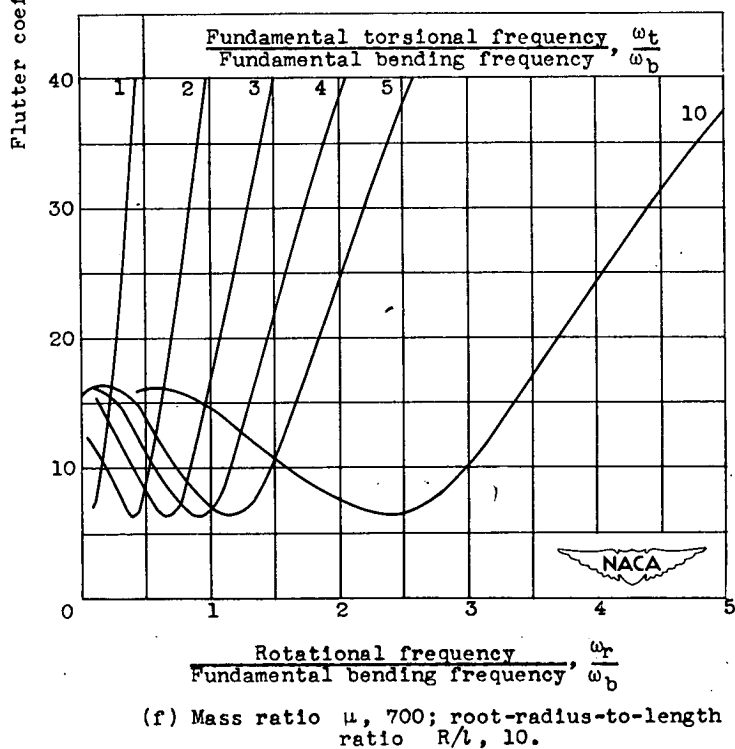
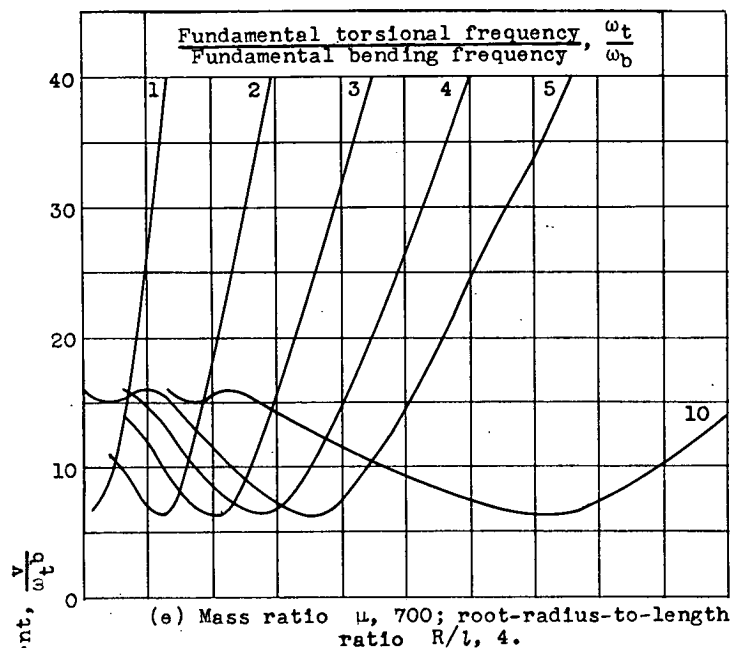


Figure 4. - Concluded. Variation of flutter coefficient with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Center-of-gravity position measured from position a , r , 0.1; elastic-axis position measured from midchord a , -0.4; radius of gyration r_g , 0.5.

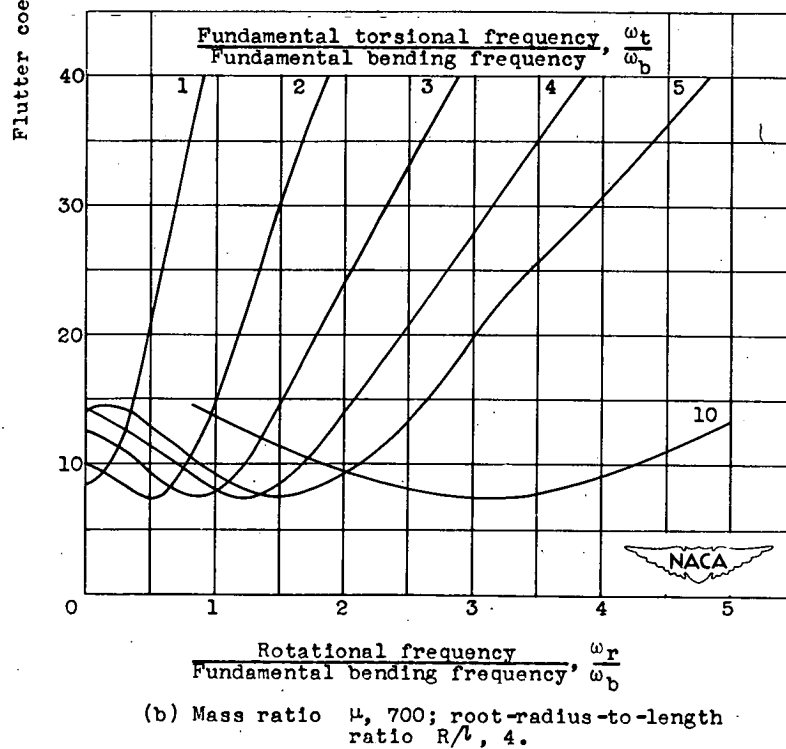
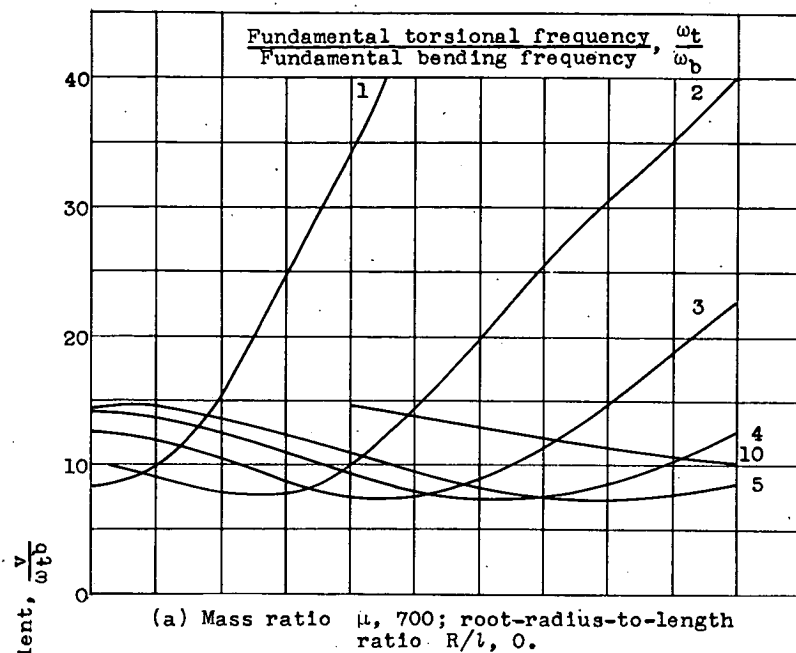
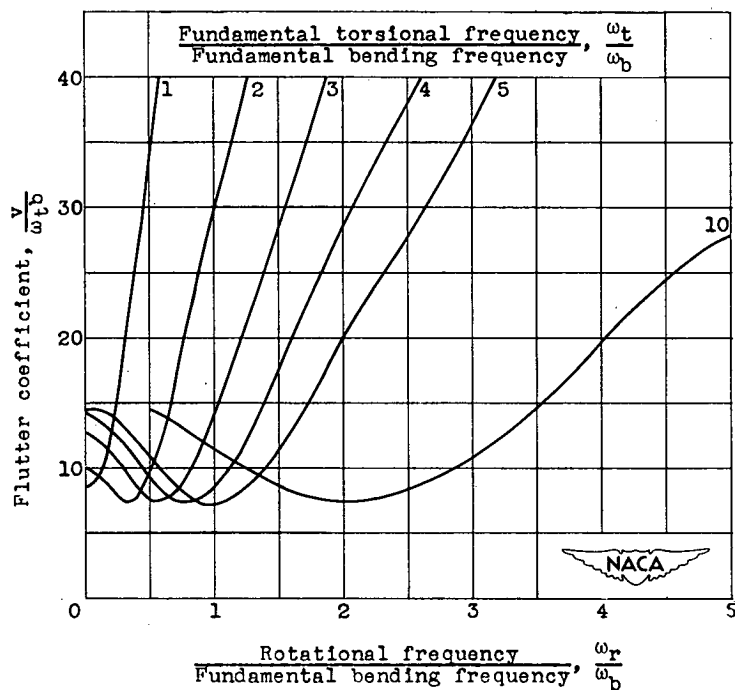


Figure 5. - Variation of flutter coefficient with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Center-of-gravity position measured from position a, r , 0.2; elastic-axis position measured from midchord a, -0.4; radius of gyration r_g , 0.5.



(c) Mass ratio μ , 700; root-radius-to-length ratio R/l , 10.

Figure 5. - Concluded. Variation of flutter coefficient with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Center-of-gravity position measured from position a , r , 0.2; elastic-axis position measured from midchord a , -0.4; radius of gyration r_g , 0.5.

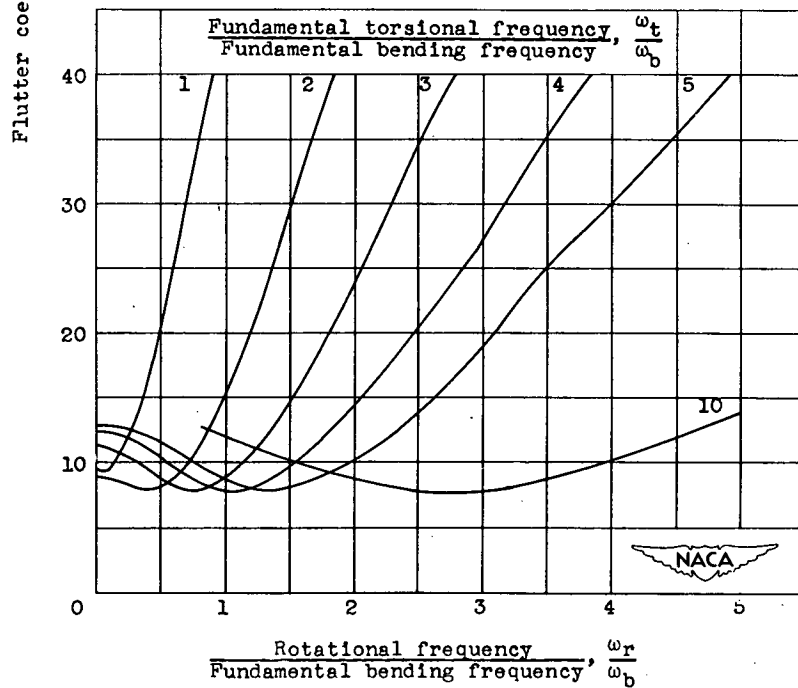
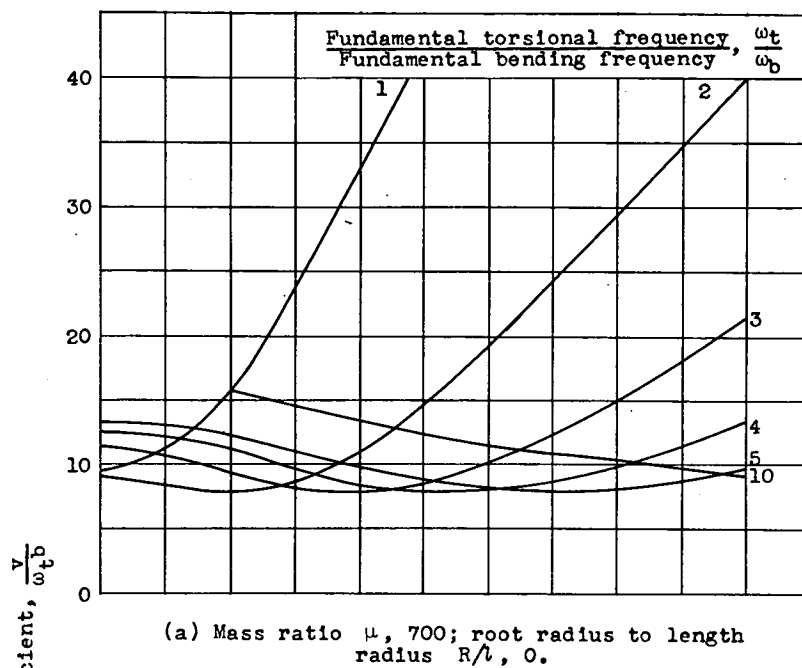
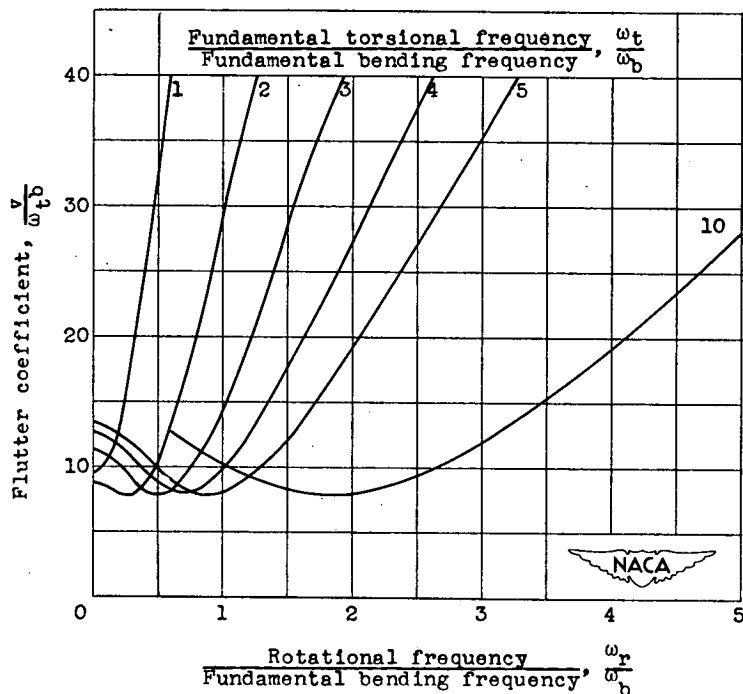


Figure 6. - Variation of flutter coefficient with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Center-of-gravity position measured from position a, r , 0.3; elastic-axis position measured from midchord a, -0.4 ; radius of gyration r_g , 0.5.



(c) Mass ratio μ , 700; root-radius-to-length ratio R/l , 10.

Figure 6. - Concluded. Variation of flutter coefficient with ratio of rotational frequency to fundamental bending frequency for various ratios of fundamental torsional frequency to fundamental bending frequency. Center-of-gravity position measured from position a, r , 0.3; elastic-axis position measured from midchord a, -0.4; radius of gyration r_g , 0.5.

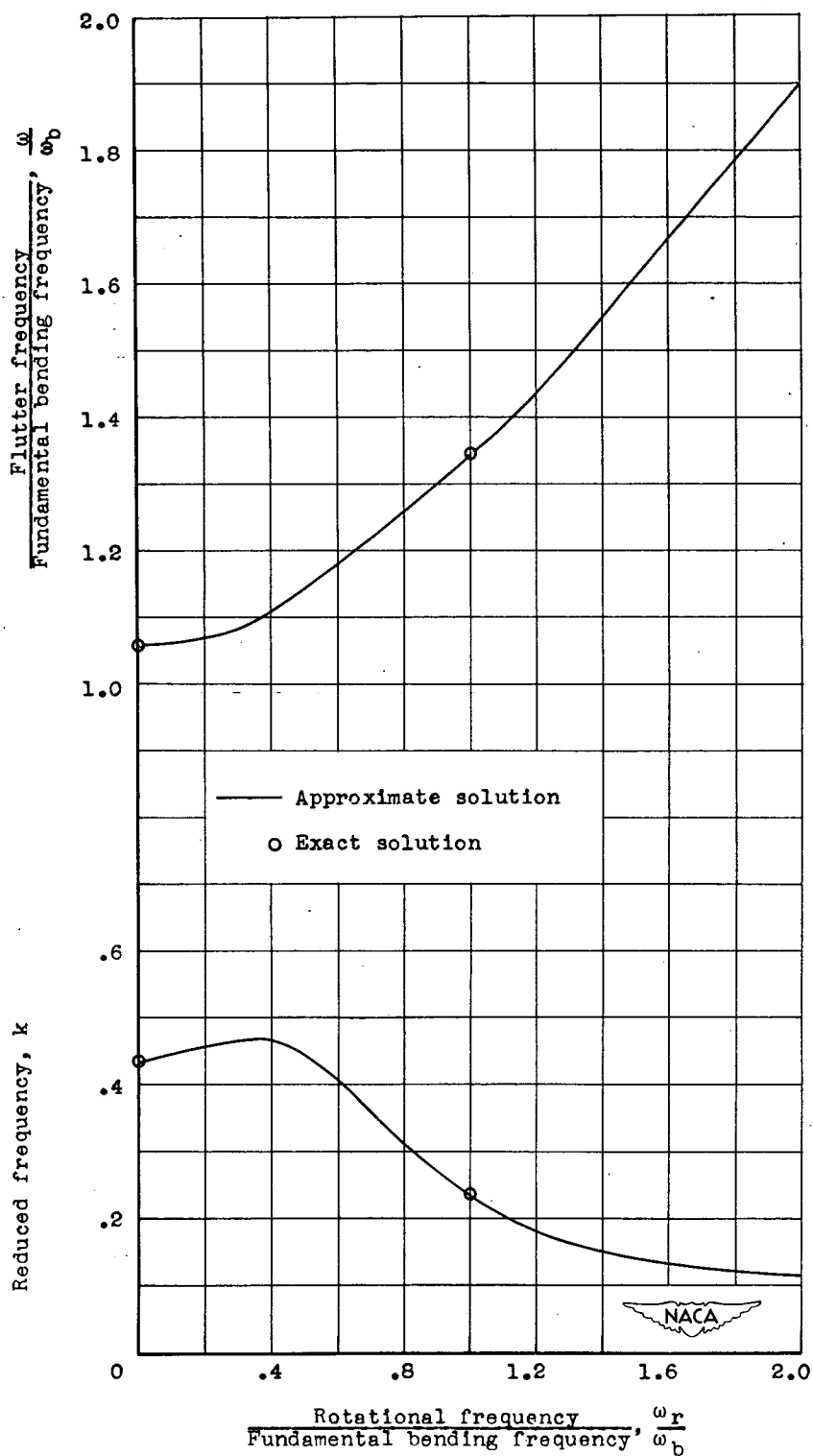


Figure 7. - Variation of ratio of flutter frequency to torsional frequency and reduced frequency with ratio of rotational frequency to fundamental bending frequency. Ratio of fundamental torsional frequency to fundamental bending frequency, 1.